Public Key Cryptography

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Asymmetric Encryption

Why Public Key Crypto is Cool

- Has a linear solution to the key distribution problem
  - Symmetric crypto has an exponential solution
- Send messages to people you don't share a secret key with
  - So only they can read it
  - They know it came for you

Number Theory

Prime Numbers

- Definition: An integer whose only factors are 1 and itself
- There are an infinite number of primes
- How many primes are there?
  - Any large number n has about a 1 in $\ln(n)$ chance of being prime

Prime Number Questions*

- If everyone needs a different prime number won't we run out?
  - Approximately 10^{151} primes 512 bits (or less)
  - Atoms in the universe: 10^{77}
  - If every atom in the universe needed 1 billion primes every microsecond from the beginning of time until now you would need 10^{109} primes
    - That means there's still about 10^{151} left

- What if two people pick the same prime?
  - Odds are significantly less than the odds of your computer spontaneously combusting at the exact moment you win the lotto

Source: Applied Cryptography (Schneier)

Prime Number Questions*

- Couldn't someone create a database of all primes and use that to break public key crypto?
  - Assuming you could store 1 GB/gram, then the weight of a drive containing the 512-bit primes would exceed the Chandrasar limit and collapse into a black hole

Source: Applied Cryptography (Schneier)

Prime Factorization :

The Fundamental Theorem of Arithmetic

- All integers can be expressed as a product of (powers of) primes
  - 48 = 2 * 2 * 2 *2* 3
- Factorization is the process of finding the prime factors of a number
- This is a **hard** problem for large numbers

Greatest Common Divisor (GCD)

- A.k.a., greatest common factor
- The largest number that evenly divides two numbers
  - GCD (15, 25) = 5

Relatively Prime

- Two numbers *x* and *y* are relatively prime if their GCD = 1
- No common factors except 1
- Example – 38 and 55 are relatively prime
  - 38 = 2 * 19
  - 55 = 5 * 11

Modular (%) Arithmetic

- Sometimes referred to as
  - "clock arithmetic" or
  - "arithmetic on a circle"
- Two numbers a and b are said to be congruent (equal) modulo N iff N/(a-b)
  - Their difference is divisible by N with no remainder
  - Their difference is a multiple of N
  - $a\%n \equiv b\%n$
  - Example – 30 and 40 are congruent mod 10
- Modulo operation
  - Find the remainder (residue) 15 **mod** 10 = 5

Notation

- *Z* - the set of integers {…-2,-1,0,1,2…}
- Z_n - the set of integers modulo *n*; {0..n-1}
- Z_n^* - the multiplicative group of integers modulo *n*
  - the set of integers modulo *n* that are relatively prime to *n*
  - Z_n^* is closed under multiplication mod *n*
  - Z_n^* does not contain 0 since the GCD(0,n)=n
  - Z_{10}^* = ?
  - Z_{12}^* = ?
  - Z_{14}^* = ?

Additive Inverse

- In Z, the additive inverse of 3 is -3,
  since 3 + -3 = 0, the additive identity.

- In Z_n, the additive inverse of a is n-a,
  since a+(n-a) = n, which is congruent to 0 (mod n).
  - What is the additive inverse of 4 mod 10?

Multiplicative Inverse

- In Z, the multiplicative inverse of 3 is 1/3, since 3*1/3=1
- The multiplicative identity in both Z and Z_n is 1
- The multiplicative inverse of 3 mod 10 is 7, since 3*7=21=1 (mod 10)
  - This could be written 3^{-1}, or (rarely) 1/3

Distributive Property

- Distribution in + and *
- Modular arithmetic is distributive.

```
a+b (mod n) =  (a mod n) + (b mod n) (mod n)
```

Proof of Distributive Property

- Let a=cn+d. Then a%n=d, the remainder after taking out the multiples of n.
- Let b=en+f. Then b%n = f.

```
   a      + b     (mod n
= cn+d + en+f (mod n)

but cn = en = 0 (mod n) (since c and
e are multiples of n), so:

= d    + f    (mod n)
```

- `= a%n + b%n (mod n).`
-

Proof of Distributive Property

- The modulus also distributes into multiplication. Consider a*b%n.

```
Let a=cn+d and b=en+f, just as before.

    a      *   b                  (mod n)
=  (cn+d)  *  (en+f)              (mod n)
=   cnen + cnf + den + df (mod n)
=  (cen)n + (cf)n + (de)n + df (mod n)

But any multiple of n modulo n is 0, so
= 0 + 0 + 0 + df   (mod n)
= a%n * b%n        (mod n)
```

Proof of Distributive Property

- An example helps:

```
  7             * 26                      (mod 5)
= (1*5 + 2) * (5*5 + 1)                   (mod 5)
= 1*5*5*5 + 1*5*1 + 2*5*5 + 2*1 (mod 5)
= 0         + 0       + 0       + 2*1 (mod 5)
= 7%5 * 26%5                              (mod 5)
= 2                                       (mod 5)
```

Big Examples

What is the sum of these numbers modulo 20?

```
   1325104987134069812734109243861723406983176
   1346139046817340961834764359873409884750983
   3632462309486723465794078340898340923876314
   3641346983862309587235093857324095683753245
+  2346982743069384673469268723406982374936877
```

Big Examples

What is the product of these numbers modulo 25?

```
    1234659823572938572
    2139582753931306947
    1398173619384713413
    2496827464249812355
    2436781359183781379
*   1351839761361377050
```

Modular Exponentiation

- Problems of the form $c = b^e \mod m$ given base b, exponent e, and modulus m
- If b, e, and m are non-negative and $b < m$, then a unique solution c exists and has the property $0 \le c < m$
- For example, $12 = 5^2 \bmod 13$
- Modular exponentiation problems are easy to solve, even for very large numbers
- However, solving the discrete logarithm (finding e given c, b, and m) is believed to be difficult

Brute Force Method

- The most straightforward method to calculating a modular exponent is to calculate b^e directly, then to take this number modulo *m*. Consider trying to compute *c*, given *b* = 4, *e* = 13, and *m* = 497:
  - One could use a calculator to compute 4^{13}; this comes out to 67,108,864. Taking this value modulo 497, the answer *c* is determined to be 445.
  - Note that *b* is only one digit in length and that *e* is only two digits in length, but the value b^e is 10 digits in length.

- In strong cryptography, *b* is often at least 256 binary digits (77 decimal digits). Consider $b = 5 * 10^{76}$ and *e* = 17, both of which are perfectly reasonable values. In this example, *b* is 77 digits in length and *e* is 2 digits in length, but the value b^e is 1304 decimal digits in length. Such calculations are possible on modern computers, but the sheer enormity of such numbers causes the speed of calculations to slow considerably. As *b* and *e* increase even further to provide better security, the value b^e becomes unwieldy.

- The time required to perform the exponentiation depends on the operating environment and the processor. If exponentiation is performed as a series of multiplications, then this requires O(*e*) time to complete.

Source: wikipedia – modular exponentiation

Diffie Hellman Project

- Write your own modular exponentiation routine
  - Use a bignum library
  - Divide and conquer algorithm O(log e)

Diffie-Hellman Key Exchange

Diffie-Hellman Key Exchange

- Allows two users to establish a secret key over an insecure medium without any prior secrets
- Two system parameters *p* and *g*.
  - Public values that may be used by all the users in a system
  - Parameter *p* is a large prime number
  - Parameter *g* (usually called a generator) is an integer less than *p*, such that for every number *n* with $0 < n < p$, there is a power *k* of *g* such that $n = g^k \bmod p$

g is primitive root

Diffie-Hellman Key Exchange

- Suppose Alice and Bob want to establish a shared secret key
- Alice and Bob agree on or use public values p,g
  - p is a large prime number
  - g is a generator
- Alice generates a random private value *a* and Bob generates a random private value *b* where *a* and *b* are integers
- Alice and Bob derive their public values using parameters *p* and *g* and their private values
  - Alice's public value = $g^a \bmod p$
  - Bob's public value is $g^b \bmod p$
- Alice and Bob exchange their public values
- Alice computes $g^{ba} = (g^b)^a \bmod p$
  Bob computes $g^{ab} = (g^a)^b \bmod p$
- Since $g^{ab} = g^{ba} = k$, Alice and Bob now have a shared secret key *k*

A Crowded Room of Mathematicians

a=50

5^{50} % 47 = 14

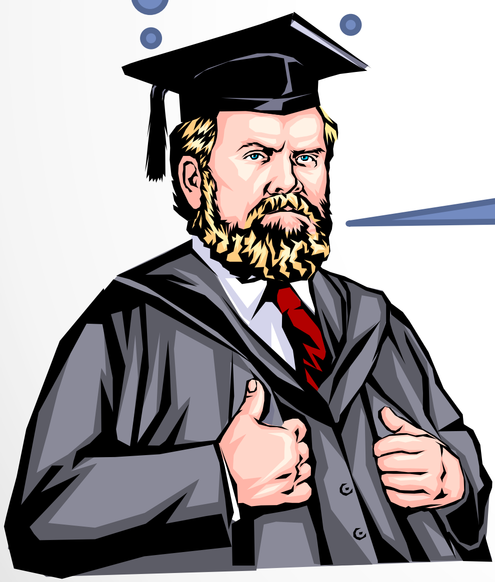
31^{50} % 47 = 18

g=5
P=47

b=49

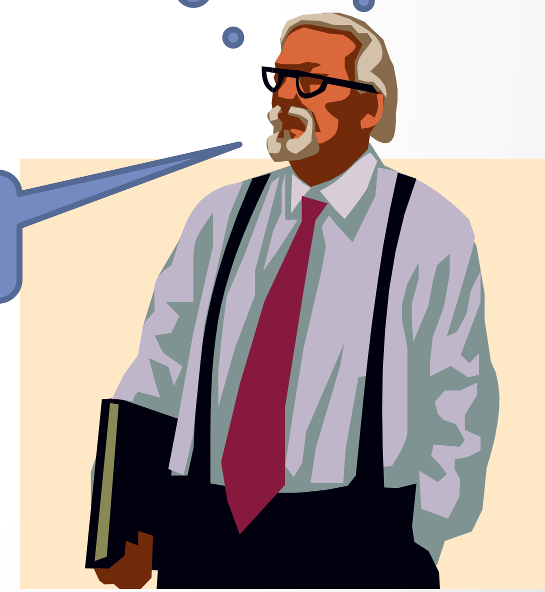
5^{49} % 47 = 31

14^{49} % 47 = 18



14

31



Why is DH Secure?

- Discrete logarithm problem
  - Inverse of modular exponentiation

- $c = b^e \bmod m$
  - e is called the “discrete logarithm”

- Solving the discrete logarithm (finding *e* given *c*, *b*, and *m*) is believed to be difficult

Attacks Against DH

- Diffie-Hellman Key Exchange is secure against a passive attacker
- How can an active attacker disrupt the protocol?
  - Man in the middle
    - Modify Alice/Bob public values as they are exchanged
      - Replace with Eve's public values
      - Replace with the value 1
      - Replace with h, where h has a small order

Practical Considerations

- Chose a safe prime p where p=2q+1 where q is also prime
- How big should p be?
  - 2048 bits (Source: Cryptography Engineering, Ferguson et al.)
  - Use p, q, and g for performance reasons (smaller subgroup)
  - Check public values for security properties
    - Public values not equal to 1
    - Public values that do not belong in too small a group
    - Hash final result of DH to generate a shared key for Alice/Bob
- How to fortify the protocol against active attackers?
  - Create a certified list of public values
  - Use digitally signed public parameters